Families of Hopf algebras of trees and pre-Lie algebras

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Abstract

Using methods from [10], we study families of Hopf algebra structures on coloured trees.

1 Introduction

Over the past years more and more examples of combinatorial Hopf algebras appeared in the mathematical literature (cf. Kreimer [7], Connes-Kreimer [3], Loday-Ronco [8], Brouder-Frabetti [1]). In most cases these Hopf algebras are constructed one at a time. One of the authors [10] constructs such Hopf algebras in families stemming from Hopf operads, rather than as isolated examples. For example, the Connes-Kreimer Hopf algebra of rooted trees and its planar analogon (cf. Foissy[4] for a detailed account) are examples related by a change of operad (see [10]). The non-planar version corresponds to a particular coproduct based on the commutative operad, whereas the planar version corresponds to its analogue for the associative operad. For the construction of some other previously studied Hopf algebras of trees in the framework of this paper we refer to [11].

The paper starts (Section 2) with the definition of the initial pair of a commutative algebra together with an n-ary map (C_n, λ_n) and applies the arguments of [10] to show that there exists a family of Hopf algebra structures on C_n. Section 3 identifies C_n with the symmetric algebra on rooted trees with n-coloured edges. Consequently the Hopf algebras constructed in Section 2 are bialgebras of trees. Section 4 then derives an explicit formula for the corresponding coproducts in terms of trees. Section 5 gives an explicit description of the Lie algebra of primitive elements of the dual Hopf algebra and gives a criterion when this is the associated Lie algebra of a pre-Lie algebra. Section 6 interprets the family of coproducts in terms of deformation theory. Finally, Section 7 sketches the more general framework of Hopf operads and lists the results obtained when one starts from associative instead of commutative algebras.

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2 Hopf algebras structures from initial objects

For definiteness, we work in the category of vector spaces over a field k of characteristic zero. (However, with the exception of Sections 5 an 6, our arguments apply in a much more general context of modules over an algebra in any symmetric monoidal additive category.) By algebra we will always mean associative algebra with unit.

2.1 Definition A *commutative n-algebra* is a pair (A, α), consisting of a commutative algebra A and a linear map $\alpha : A^{\otimes n} \longrightarrow A$. A morphism of n-algebras $f : (A, \alpha) \longrightarrow (B, \beta)$ is an algebra homomorphism $f : A \longrightarrow B$ such that $\beta \circ f^{\otimes n} = f \circ \alpha$. We will write $\mathcal{C}_n$ for the category of these n-algebras. For general reasons, this category $\mathcal{C}_n$ has an initial object, the free n-algebra on the empty set of generators. This initial algebra will be denoted by (C_n, λ_n). It is completely characterised (up to isomorphism) by the property that any n-algebra (A, α) admits a unique morphism $(C_n, \lambda_n) \longrightarrow (A, \alpha)$. We will give an explicit description of this initial algebra in Proposition 3.2 below.

This section introduces Hopf algebra structures on C_n of a specific type. First we need a notation. If $(A, \alpha) \in \mathcal{C}_n$ is a commutative n-algebra and $\sigma_1, \sigma_2 : A^{\otimes n} \longrightarrow A$ are two linear maps, define the linear map $(\sigma_1, \sigma_2) : (A \otimes A)^{\otimes n} \longrightarrow A \otimes A$ as

$$(\sigma_1, \sigma_2) = (\sigma_1 \otimes \alpha + \alpha \otimes \sigma_2) \circ \tau.$$

(Here $\tau : (A^{\otimes 2})^{\otimes n} \longrightarrow A^{\otimes n} \otimes A^{\otimes n}$ is the isomorphism that separates the first and second tensor factors from $A^{\otimes 2}$.)

We are going to use this in the context where (A, α) is the initial n-algebra (C_n, λ_n). First note that (C_n, λ_n) is an augmented algebra. Indeed, the ground field k is naturally an n-algebra when equipped with the zero map $k^{\otimes n} \longrightarrow k$. So by initiality of (C_n, λ_n), there is a unique morphism of n-algebras $\varepsilon : (C_n, \lambda) \longrightarrow (k, 0)$.

For any pair of n-ary linear maps $\sigma_1, \sigma_2 : C_n^{\otimes n} \longrightarrow C_n$ as above there is a unique morphism $\Delta : (C_n, \lambda_n) \longrightarrow (C_n \otimes C_n, (\sigma_1, \sigma_2))$ in $\mathcal{C}_n$. That is, a unique algebra morphism Δ such that the diagram

$$\begin{array}{ccc} C_n^{\otimes n} & \xrightarrow{\lambda} & C_n \\ \downarrow{\scriptstyle \Delta^{\otimes n}} & & \downarrow{\scriptstyle \Delta} \\ (C_n \otimes C_n)^{\otimes n} & \xrightarrow{(\sigma_1, \sigma_2)} & C_n \otimes C_n \end{array}$$

commutes.

2.2 THEOREM *Let $n \in \mathbb{N}$, λ_n and C_n be defined as above. Let $\sigma_i : C_n^{\otimes n} \longrightarrow C_n$ for $i = 1, 2$ be linear maps. If both σ_i satisfy*

$$\epsilon \circ \sigma_i = \epsilon^{\otimes n}, \quad \text{and}$$
$$\Delta \circ \sigma_i = (\sigma_i \otimes \sigma_i) \circ \tau \circ \Delta^{\otimes n};$$

then there exists a unique bialgebra structure on C_n such that $\Delta \circ \lambda = (\sigma_1, \sigma_2) \circ \Delta^{\otimes n}$ and $\varepsilon \circ \lambda = 0$.

PROOF The proof that this provides a bialgebra structure is completely analogous to the case where $n = 1$, treated in [10]. QED

2.3 REMARK We will see in the next section that the initial algebra C_n has a natural grading. The coalgebra structure given by a pair of maps σ_1 and σ_2 will respect this grading if these maps σ_1 and σ_2 do. In this case there exists an antipode for the bialgebra structure. A special case of this occurs in Theorems 4.2 and 4.4 below.

2.4 REMARK The algebras C_n together form a simplicial algebra. The simplicial operations $d_i : C_n \longrightarrow C_{n-1}$ and $s_i : C_n \longrightarrow C_{n+1}$ are the algebra homomorphisms determined by

$$\begin{aligned}
d_0(\lambda_n)(x_1, \ldots, x_n) &= \mu(x_1, \lambda_{n-1}(x_2, \ldots, x_n)) \\
d_i(\lambda_n)(x_1, \ldots, x_n) &= \lambda_n(x_1, \ldots, \mu(x_i, x_{i+1}), \ldots, x_n) \qquad (i = 1, \ldots, n-1) \\
d_n(\lambda_n)(x_1, \ldots, x_n) &= \mu(\lambda(x_1, \ldots, x_{n-1}), x_n) \\
s_i(\lambda_n)(x_1, \ldots, x_n) &= \lambda_{n+1}(x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_n) \qquad (i = 0, \ldots, n),
\end{aligned}$$

similar to the formulas for the Hochschild complex.

3 Initial algebras and trees

3.1 DEFINITION A *rooted tree* is an isomorphism class t of finite partially ordered sets which

(i). have a minimal element r ($\forall x \neq r : r < x$), called the *root*, and

(ii). satisfy the tree condition that $(y \neq z) \wedge (y < x) \wedge (z < x)$ implies $(y < z) \vee (z < y)$.

In general, we will not be very precise in distinguishing between an isomorphism class t and any of the posets which represent it; in particular, we will often use t to denote to a representing poset, and refer to it as a tree.

The elements of a tree are called *vertices*. A pair of vertices $v < w$ is called an *edge* if there is no vertex x such that $v < x < w$. The number of vertices of a tree t is denoted by $|t|$. A *path* from x to y in a tree is a sequence $(x_i)_i$ of elements $x = x_n > x_{n-1} > \ldots > x_1 > x_0 = y$ of maximal length. We will say that x is above y in a tree if there is a path from x to y.

A *forest* is a finite (possibly empty) multiset (i.e. a set with multiplicities) of trees. A subforest of a tree or forest is a subset of vertices with the induced partial order.

In the sequel we need *trees with coloured edges*. These are isomorphism classes of posets as above, equipped with a function from the set of edges to a fixed set of colours. The isomorphisms are required to respect the colours. In particular, an *n-coloured tree* is such a tree whose edges are coloured by the set $\{1,..,n\}$ of colours. We will write T_n for the vector space spanned by the set of such n-coloured trees.

3.2 Proposition *There is a natural algebra isomorphism between the initial n-algebra C_n and the symmetric algebra $S(T_n)$ on the set of n-coloured trees.*

Proof The symmetric algebra $S(T_n)$ can be identifyed with the vector space spanned by the set of n-coloured forests, with the unit represented by the empty forest and the product by the disjoint union of forests. There is an operation

$$\lambda : S(T_n)^{\otimes n} \longrightarrow S(T_n)$$

which takes an n-tuple of n-coloured forests $f_1, \ldots, f_n$, and combines them into a single n-coloured tree by adding a new root, while connecting this new root to each of the roots in the forest f_i by an edge of colour i.

This operation makes $S(T_n)$ into an object of the category $\mathcal{C}_n$. Since initial objects are unique up to isomorphism in any category, it now suffices to prove that $S(T_n)$ is initial in $\mathcal{C}_n$.

To this end, let (A, α) be any object of $\mathcal{C}_n$, where $\alpha : A^{\otimes n} \longrightarrow A$. Define a morphism

$$\varphi : (S(T_n), \lambda) \longrightarrow (A, \alpha)$$

by induction on trees and forests. If $f = t_1 \cdot \ldots \cdot t_k$ is a forest consisting of k trees, then $\varphi(f) = \varphi(t_1) \cdot \ldots \cdot \varphi(t_k)$, so it suffices to define φ on trees. If t is a tree consisting of a root only, then $\varphi(t) = \alpha(1, \ldots, 1)$. If t consists of a root r onto which an n-tuple of n-coloured forests $f_1, \ldots, f_n$ is attached by joining the root of each tree in f_i to r via an edge of colour i, then $\varphi(t) = \alpha(\varphi(f_1), \ldots, \varphi(f_n))$. It is straightforward to check that $\varphi : (S(T_n), \lambda) \longrightarrow (A, \alpha)$ thus defined is indeed a morphism in $\mathcal{C}_n$, and is the unique such. QED

4 Hopf algebras of trees

In this section, we study a particular example of a family of Hopf algebras which can be obtained by the general method of Theorem 2.2.

4.1 Example For any choice of $q_{i\,j} \in k$ for all $j \leq n$, the maps

$$\sigma_i(t_1, \ldots, t_n) = q_{i\,1}^{|t_1|} \cdot \ldots \cdot q_{i\,n}^{|t_n|} t_1 \cdot \ldots \cdot t_n \qquad \text{for } i = 1, 2$$

define a bialgebra structure on $S(T_n)$. Here and in the sequel $|f|$ denotes the number of vertices in the forest corresponding to $f \in S(T_n)$, while the associative multiplication on $S(T_n)$ is denoted by $\cdot$.

Below we write $\Delta(f_i) = \sum f_i' \otimes f_i''$ reminiscent of the form Δ takes in a basis.

4.2 THEOREM *The symmetric algebra $S(T_n)$ on n-coloured trees has a natural family of graded connected Hopf algebra structures, indexed by sequences $(q_{11}, \ldots, q_{1n}, q_{21}, \ldots, q_{2n}) \in k^{2n}$. The grading is with respect to the number of vertices of the trees. An inductive description of the coproduct is given by*

$$\begin{aligned}\Delta(\lambda(f_1, \ldots, f_n)) = &\sum q_{11}^{|f_1'|} \cdot \ldots \cdot q_{1n}^{|f_n'|} \cdot f_1' \cdot \ldots \cdot f_n' \otimes \lambda(f_1'', \ldots, f_n'') \\ &+ \sum \lambda(f_1', \ldots, f_n') \otimes q_{21}^{|f_1''|} \cdot \ldots \cdot q_{2n}^{|f_n''|} \cdot f_1'' \cdot \ldots \cdot f_n'',\end{aligned}$$

where $\lambda(f_1, \ldots, f_n)$ is the rooted tree obtained for n forests $f_1, \ldots, f_n$ by adding a new root and connecting each of the roots of trees in f_i to the new root by an edge of colour i, and where $|f_i|$ is the number of vertices in the forest f_i.

PROOF The bialgebra structures are a direct translation of Example 4.1 to the language of trees of Proposition 3.2. The bialgebra $S(T_n)$ is graded connected with respect to the grading $|.|$. It is well known (cf. Milnor and Moore [9]) that any graded connected bialgebra admits an antipode. QED

We now turn to the question of finding a more direct decription of these Hopf algebra structures. Fix $n \in \mathbb{N}$. For $i = 1, 2$ and $1 \leq j \leq n$, let $q_{ij} \in k$, and define for $t_i \in S(T_n)$

$$\sigma_i(t_1, \ldots t_n) = (\prod_j q_{ij}^{|t_j|}) \cdot t_1 \cdot \ldots \cdot t_n. \tag{4.1}$$

Any rooted tree has a natural partial ordering on its vertices in which the root is the minimal element. A *subforest* s of a rooted tree t is a subset of the partially ordered set (representing) t with the induced partial ordering. For $v \in s$ we denote by $p_k(v, s, t)$ the number of edges of colour k in the path in t from v to the root of t that have their lower vertex in s^c. For forests t we define $p_k(v, s, t)$ as $p_k(v, s \cap t', t')$, where t' is the connected component of t containing v. There is an easy but useful lemma on the calculus of the p_k.

4.3 LEMMA *Let t and s be subforests of a forest u. Let $v \in s$ and set $t' = t \cup v$, $s' = s \cap t'$, $t'' = t^c \cup v$ and $s'' = s \cap t''$. Then*

$$p_k(v, s, u) = p_k(v, s', t') + p_k(v, s'', t''),$$

where t', t'', s' and s'' are interpreted as subforests of u.

Proof The lemma follows at once when we observe that a vertex in the path from v to the root in u that is not in s is either in t' or in t''. QED

Define for $s \subset t$ a subforest

$$q(s,t) := \prod_j \left(\prod_{v \in s} q_{1j}^{p_j(v,s,t)} \cdot \prod_{v \in s^c} q_{2j}^{p_j(v,s^c,t)} \right). \tag{4.2}$$

More intuitively, $q(s,t)$ counts for $v \in s$ the number of edges of colour j in the path from v to the root that have their lower vertex in s^c and adds a factor q_{1j} for each of these, and $q(s,t)$ counts for $v \in s^c$ the number of edges of colour j in the path from v to the root that have their lower vertex in s and adds a factor $q_{2,j}$ for each of these.

4.4 Theorem *Let $S(T_n)$ be the symmetric algebra on n-coloured trees as in Theorem 4.2.*

(i). For a forest $t \in S(T_n)$ the coproduct defined by $(q_{11}, \ldots, q_{2n}) \in k^{2n}$ is given by the formula

$$\Delta(t) = \sum_{s \subset t} q(s,t)\, s \otimes s^c,$$

where the sum is over all subforests s of t.

(ii). The antipode of the Hopf algebra $S(T_n)$ with the coproduct of part (i) is given by

$$S(t) = \sum_{k=1}^{|t|} \sum_{\cup_i s_i = t} (-1)^k s_1 \cdot \ldots \cdot s_k \prod_{1 \le j < k} q(s_j, s_j \cup \ldots \cup s_k),$$

where we only sum over (ordered) partitions $t = s_1 \cup \ldots \cup s_k$ of the forest t with all forests s_i non-empty.

Proof We use induction with respect to the number of applications of λ to show the first result. The formula is trivial for the empty tree. Let $t = \lambda(x_1, \ldots, x_n)$ be a tree and suppose (as the induction hypothesis) that the formula holds for all trees with less then $|t|$ vertices. Since Δ is an algebra morphism, it is clear the formula also holds for the forests x_i since these consist of trees with less than $|t|$ vertices. Subforests of t are either of the form $s = \cup_i s_i$, a (disjoint) union of subforests of the x_i, or of the form $s = r \cup (\cup_i s_i)$, a (disjoint) union of subforests of the x_i together with the root. By definition and the induction hypothesis,

$$\begin{aligned}\Delta(t) = &\sum_{s_i \subset x_i} s_1 \cdot \ldots \cdot s_n \otimes \lambda(s_1^c, \ldots, s_n^c) \cdot \prod_i q_{1i}^{|s_i|} q(s_i, x_i) \\ &+ \sum_{s_i \subset x_i} \lambda(s_1, \ldots, s_n) \otimes s_1^c \cdot \ldots \cdot s_n^c \cdot \prod_i q_{2i}^{|s_i^c|} q(s_i, x_i).\end{aligned}$$

But by the lemma above,

$$\prod q_{1i}^{|s_i|} q(s_i, x_i) = \prod_j \left(\prod_{v \in s} q_{1j}^{p_j(v,s,t)} \cdot \prod_{v \in s^c} q_{2j}^{p_j(v,s^c,t)} \right)$$

for $s = \cup_i s_i = s_1 \cdot \ldots \cdot s_n$ and $s^c = r \cup (\cup_i s_i^c) = \lambda(s_1^c, \ldots, s_n^c)$; and

$$\prod q_{2i}^{|s_i^c|} q(s_i, x_i) = \prod_j \left(\prod_{v \in s} q_{1\,j}^{p_j(v,s,t)} \cdot \prod_{v \in s^c} q_{2\,j}^{p_j(v,s^c,t)} \right)$$

for $s = r \cup (\cup_i s_i) = \lambda(s_1, \ldots, s_n)$ and $s^c = \cup_i s_i = s_1^c \cdot \ldots \cdot s_n^c$. Putting these together proves the formula for the coproduct.

Let A be a graded connected bialgebra. The augmentation ideal of A is $\bar{A} = \bigoplus_{n \geq 1} A^n$. The antipode on A applied to $x \in \bar{A}$ is given by

$$S(x) = \sum_{k=0}^{\infty} (-1)^{k+1} \mu^{(k)} \circ \bar{\Delta}^{(k)}(x),$$

where $\bar{\Delta} = \Delta - (\mathrm{id} \otimes 1 + 1 \otimes \mathrm{id})$, and $\mu^{(0)} = \mathrm{id} = \bar{\Delta}^{(0)}$, and $\mu^{(k)} : A^{\otimes k+1} \longrightarrow A$ and $\bar{\Delta}^{(k)} : A \longrightarrow A^{\otimes k+1}$ are defined using (co)associativity for $k > 0$ and $\mu^{(0)} = \mathrm{id} = \bar{\Delta}^{(0)}$. (The sum in the formula for $S(x)$ is of course finite, and stops at k for a homogeneous element x of degree k.) As a special case, consider $S(T_n)$ with the coproduct just described. This proves the result. QED

4.5 Example The Hopf algebra C_1 with the coproduct defined by $q_{11} = 1$ and $q_{21} = 0$ is the Connes-Kreimer Hopf algebra of rooted trees [7, 3, 10].

5 Primitives of the dual

Since $S(T_n)$ is commutative, we know by the Milnor-Moore Theorem [9] that the graded linear dual $S(T_n)^*$ is the universal enveloping algebra of the Lie algebra of its primitive elements. The result below provides an explicit formula for the Lie bracket on the these primitive elements.

5.1 Corollary *Let $S(T_n)$ be a the symmetric algebra on rooted trees with n-coloured edges, and let Δ be the coproduct defined by $(q_{11}, \ldots, q_{2n}) \in k^{2n}$ (cf. Theorem 4.4). The graded dual $S(T_n)^*$ is the universal enveloping algebra of the Lie algebra which as a vector space is spanned by elements D_t, where t is a rooted tree in $S(T_n)$. The bracket is given by $[D_s, D_t] = D_t \bullet D_s - D_s \bullet D_t$, where*

$$D_t \bullet D_s = \sum_w \sum_{s \subset w,\ s^c = t} q(s, w) D_w.$$

In this formula, the first sum ranges over all rooted trees w, and the second sum over subtrees of w which are isomorphic to s and whose complement s^c is isomorphic to t.

Proof For any cocommutative Hopf algebra we can define an operation $\bullet$ on the primitive elements, such that its commutator is the Lie bracket on primitive elements: Simply define $\bullet$ as the truncation of the

product at degree > 1, with respect to the primitive filtration F. In this case, $F_m C_n^*$ is spanned by the elements D_u dual to forests u consisting of at most m trees. The product in $S(T_n)^*$ is determined by the coproduct in $S(T_n)$. For forest u, we can write the multiplication in C_n^* as

$$(D_t D_s)(u) = (D_t \otimes D_s)\Delta(u),$$

thus, for a fixed u we get a contribution $q(u,s)D_u$ for every subtree isomorphic to s in u with t as complementary forest. When we then restrict to the primitive part (i.e. the part where u is a tree as well), we conclude that $D_t \bullet D_s$ is given by the desired formula. QED

Recall (cf. Chapoton and Livernet [2]) that a pre-Lie algebra is vector space L together with a bilinear operation $\bullet$ satisfying the identity

$$(x \bullet y) \bullet z - x \bullet (y \bullet z) = (x \bullet z) \bullet y - x \bullet (z \bullet y).$$

The the *free pre-Lie algebra* L_n on n generators is given by the vector space spanned by rooted trees with vertices labelled by elements of the set $\{1, 2, \ldots, n\}$. The pre-Lie algebra product is given by grafting trees: For t and s trees, and v a vertex in t, denote by $t \circ_v s$ the tree obtained from t and s by attaching the root of s to vertex v in t by a new edge. Grafting preserves the labeling of the vertices. The pre-Lie algebra structure on L_n is given by,

$$t \bullet s = \sum_{v \in t} t \circ_v s$$

for trees s and t.

Below we denote by χ_S the characteristic function of a subset $S \subset X$ which has value 1 on S and value 0 on $X - S$, and denote the primitive elements of a coalgebra C by $P(C)$.

5.2 Theorem *Let $\mathbf{p} \subset \{1, \ldots, n\}$ and define $q_{1j} = \chi_{\mathbf{p}}(j)$ and $q_{2j} = 0$ for $j = 1, \ldots, n$. Consider the Hopf algebra structure on the symmetric algebra $S(T_n)$ on rooted trees with n-coloured edges that corresponds to this choice of q_{ij}.*

(i). The product $D_t \bullet D_s = \sum_w \sum_{s \subset w, s^c = t} q(s,w)\, D_w$ of Corollary 5.1 defines a pre-Lie algebra structure on the primitive elements $P(C_n^*)$ of $S(T_n)^*$.

(ii). If $\mathbf{p} = \{1, \ldots, n\}$, then there is a natural inclusion of this pre-Lie algebra into the free pre-Lie algebra on n generators. The image in the free pre-Lie algebra is spanned by all sums $\sum_{i \in \mathbf{p}} t_i$, of trees with vertices coloured by $\mathbf{p}$ that only differ in that the colour of the root of t_i is i.

Proof Consider the general formula for $D_t \bullet D_s$ in Corollary 5.1. Note that for $q_{ij} \in \{0, 1\}$ the coefficients $q(s,w)$ are either 0 or 1. We can be more precise. Let w and t be trees. A product of subtrees $s = s_1 \cdot \ldots \cdot s_m \subset w$ is t-admissible if s^c contains the root of w while s is grafted onto $t = s^c$

by a edges of colours $i_1, \dots, i_m \in \mathbf{p}$ to vertices $v_1, \dots, v_m$ respectively each of which is connected to the root by edges having colours in $\mathbf{p}$. We only use this terminology for $n = 1, 2$. Thus, in this case we consider $q(s, w) = 0$ unless the the corresponding subtree s is t-admissible.

The pre-Lie identity follows from

$$(D_t \bullet D_s) \bullet D_u - D_t \bullet (D_s \bullet D_u) = \sum_w \sum_{s \cdot u \subset w} D_w,$$

where the second sum is over t-admissible products $s \cdot u$. This proves part *(i)* since the expression is symmetric in s and u.

Before we prove *(ii)* we study the pre-Lie algebra structure $\bullet$ is a bit more detail. Let $m(t, s, w)$ be the number of t-admissible subtrees $s \subset w$. The operation $\bullet$ is then given by

$$D_t \bullet D_s = \sum_w m(t, s, w) D_w,$$

where the sum is over all rooted trees. For our aims it is better to use a different description of this pre-Lie algebra. We closely follow the strategy of Hoffman [6] in this respect. If $m(t, s, w) \neq 0$, it is exactly the order of the orbit of the root of the subtree s under the action of the group $\mathrm{Aut}(w)$ of automorphisms of w. If s and t are n-coloured trees and v is a vertex in t, denote by $t \circ_{(v,i)} s$ the tree obtained from t and s by connecting the root of s to the vertex v by an edge of colour i. Let $n(t, s, w)$ be the number if vertices $v \in t$ such that $t \circ_{(v,i)} s = w$ for some $i \in \mathbf{p}$. Then for any such vertex v, the order of the orbit of v in t under the action of $\mathrm{Aut}(t)$ is exactly $n(t, s, w)$.

Define an other pre-Lie algebra structure $\bullet'$ on the same vector space $P(C_n^*)$, by

$$D_t \bullet' D_s = \sum_w n(t, s, w)\, D_w,$$

and denote this pre-Lie algebra by $P(C_n^*)'$. For a subtree $s \subset w$, denote by $\mathrm{Aut}^s(w)$ the automorphisms of w that pointwise fix s. Then, if $m(s, t, w) \neq 0$ we can write, following Hoffman [6],

$$\begin{aligned} m(t, s, w) &= \frac{|\mathrm{Aut}(w)|}{|\mathrm{Aut}^s(w)| \cdot |\mathrm{Aut}(t)|} \\ n(t, s, w) &= \frac{|\mathrm{Aut}(s)|}{|\mathrm{Aut}^{\{v\}}(t)|}, \end{aligned}$$

for a vertex v such that $t \circ_{(v,i)} s = w$ for some i. Since $|\mathrm{Aut}^s(w)| = |\mathrm{Aut}^{\{v\}}(t)|$ it follows that $D_t \longmapsto |\mathrm{Aut}(t)| D_t$ defines an isomorphism of pre-Lie algebras $P(C_n^*) \longrightarrow P(C_n^*)'$ (in characteristic 0).

In the remainder of the proof, let $\mathbf{p} = \{1, 2, \dots, n\}$. We prove *(ii)* by constructing an inclusion $P(C_n^*)' \longrightarrow L_n$. Note that

$$D_t \bullet' D_s = \sum_{v \in t} \sum_{i \in \mathbf{p}} D_{t \circ_{(v,i)} s}$$

For an n-coloured tree t, denote by $\uparrow_i(t)$ the tree with coloured vertices obtined by moving the colour of ech edge up to the vertex directly above

it and colouring the root by i. Note that

$$\uparrow_j (t \circ_{(v,i)} s) = \uparrow_i (t) \circ_v \uparrow_j (s).$$

Let $\mathbf{p} = \{1, \ldots, n\}$ and consider $S(T_n)$ with the corresponding Hopf algebra structure as defined above. Define $\varphi : P(S(T_n)^*)' \longrightarrow L_n$ from the pre-Lie algebra of primitives to the free pre-Lie algebra on n generators by

$$\varphi(D_t) = \sum_{j=1}^{n} \uparrow_j (t).$$

Then φ is a linear enbedding. Moreover, φ preserves the pre-Lie algebra structure since

$$\begin{aligned} \varphi(D_t \bullet' D_s) &= \sum_{(v,i)} \varphi(D_{t \circ_{(v,i)} s}) \\ &= \sum_{(v,i)} \sum_j \uparrow (t \circ_{(v,i)} s) \\ &= \sum_v \sum_{i,j} \uparrow_i (t) \circ_v \uparrow_j (s) \\ &= \varphi(D_t) \bullet \varphi(D_s). \end{aligned}$$

QED

5.3 EXAMPLE In the case of the Connes-Kreimer Hopf algebra (Example 4.5), Theorem 5.2 states that the dual Hopf algebra is the universal enveloping algebra of the free pre-Lie algebra on one generator as first proved by Chapton-Livernet [2].

6 Formal coalgebra deformations

In this section we consider how the different Hopf algebra structures of Theorem 4.2 are related from the point of view of deformation theory. For simplicity we restrict our attention to the case $n = 1$.

Let A be a Hopf algebra. Recall (e.g. Gerstenhaber-Shack [5]) the bicomplex $C^{p\,q}(A) = \mathrm{Hom}(A^{\otimes p}, A^{\otimes q})$ for $p, q \geq 1$. For q fixed it is the Hochschild complex of the algebra A with coefficients in $A^{\otimes q}$, and for p fixed it is the Hochschild complex of the coalgebra A with coefficients in $A^{\otimes p}$. Classes in $H^3(\mathrm{Tot}(C^{**}(A)))$ are in 1-1 correspondence with Hopf algebra deformations of A modulo $\hbar^2$.

A formal coalgebra deformation $\Delta_\hbar$ of a Hopf algebra A is a $k[[\hbar]]$-linear map $\Delta_\hbar : A[[\hbar]] \longrightarrow A[[\hbar]] \otimes_{k[[\hbar]]} A[[\hbar]$ for which $\Delta_\hbar$ makes $A[[\hbar]]$ with the same multiplication and counit a Hopf algebra over $k[[\hbar]]$, and such that evaluation at $\hbar = 0$ gives the original Hopf algebra structure on A.

The vector spaces $\mathrm{Der}(A, A^{\otimes q})$ of algebra derivations form the kernel of the horizontal differential at the edge of the complex $C^{*q}(A)$, and thus a

subcomplex of the coalgebra Hochschild complex. Coalgebra deformations modulo $\hbar^2$ of the Hopf P-algebra A are in 1-1 correspondence with classes in $H^2(\mathrm{Der}(A, A^{\otimes *}))$.

Let us now turn to the example of the Hopf algebra $S(T_1)$ with the coproduct on trees given by $\Delta(s) = s \otimes 1 + 1 \otimes s$. This is the coproduct induced by $\sigma_1 = \sigma_2 = u \circ \hbar$. If $q_1, q_2 \in t \cdot k[[\hbar]]$ and if we write $\lambda := \lambda_1$, then the map $\Delta_{q_1,q_2} : S(T_1)[[\hbar]] \longrightarrow S(T_1)[[\hbar]] \otimes_{k[[\hbar]]} S(T_1)[[\hbar]]$ inductively defined by

$$\Delta_{q_1,q_2} \circ \lambda(x) = \sum_{(x)} \lambda(x_1) \otimes q_2^{|x_2|} x_2 + q_1^{|x_1|} x_1 \otimes \lambda(x_2)$$

defines a coalgebra deformation of $S(T_1)$.

The Hopf algebra $S(T_1)$ is graded. Let us write $\mathrm{Der}_0(S(T_1), S(T_1)^{\otimes *})$ for the subcomplex of $\mathrm{Der}(S(T_1), S(T_1)^{\otimes *})$ consisting of those derivations that preserve the degree. Classes in $H^2(\mathrm{Der}_0(S(T_1), S(T_1)^{\otimes *}))$ correspond to graded coalgebra deformations. The result below studies the deformations Δ_{q_1,q_2} as graded coalgebra deformations.

6.1 Proposition *Consider $S(T_1)$ with the coproduct induced by $\sigma_1 = \sigma_2 = u \circ \hbar$.*

(i). The boundaries in $\mathrm{Der}_0(S(T_1), S(T_1)^{\otimes 2})$ are the derivations φ that can be written as

$$\varphi(s) = \sum_w c_{s,w} \bar{\Delta}(w),$$

for all trees s, and constants $c_{s,w} \in k$, and where the sum ranges over all forests w such that $|s| = |w|$. As usual, $\bar{\Delta} = \Delta - (\mathrm{id} \otimes 1 + 1 \otimes \mathrm{id})$.

(ii). Let $q_1 \equiv c_1\hbar$, $q_2 \equiv c_2\hbar$, and $q_1' \equiv d_1\hbar, q_2' \equiv d_2\hbar$ modulo $\hbar^2$. The two graded coalgebra deformations Δ_{q_1,q_2} and $\Delta_{q_1',q_2'}$ are equivalent modulo $\hbar^2$ iff $c_1 - c_2 = d_1 - d_2$.

Proof Let $\psi \in \mathrm{Der}_0(S(T_1), S(T_1))$. Then $\psi(1) = 0$, and ψ is determined by its values on trees. Write ψ in matrix form as $\psi(s) = \sum_w c_{s,w} w$, where the sum ranges over forests, and $c_{s,w} \in k$ are constants. Since we assume ψ is graded, $c_{s,w} = 0$ if $|w| \neq |s|$. Compute

$$\begin{aligned} d\psi(s) &= (\psi(s) \otimes 1 + 1 \otimes \psi(s)) - \Delta(\psi(s)) \\ &= \sum_w c_{s,w}(w \otimes 1 + 1 \otimes w) - c_{s,w}\Delta(w) \\ &= -\sum_w c_{s,w}\bar{\Delta} w. \end{aligned}$$

For $\psi : S(T_1) \longrightarrow S(T_1)$ as above define the endomorphism Ψ of $S(T_1)[[\hbar]]$ by $\Psi(x) = x + \psi(x)t$ for $x \in S(T_1)[[\hbar]]$. Two coalgebra deformations $\Delta_{q_1 q_2}$ and $\Delta_{q_1' q_2'}$ are equivalent modulo $\hbar^2$ iff we can find a derivation ψ such that the corresponding map Ψ satisfies for all s

$$\Delta_{q_1' q_2'} \circ \Psi(s) \equiv \Psi \otimes \Psi \circ \Delta_{q_1,q_2}(s), \qquad \text{or equivalently}$$

$$\Delta_{q_1' q_2'}(s + \psi(s)\hbar) \equiv \Delta_{q_1,q_2}(s) + (\psi \otimes 1)(\Delta_{q_1,q_2}(s))\hbar + (1 \otimes \psi)(\Delta_{q_1,q_2}(s))\hbar$$

modulo $\hbar^2$. To compute this we only have to consider terms in $\Delta_{q_1'q_2'}(s)$ and $\Delta_{q_1,q_2}(s)$ corresponding to $u \subset s$, such that

$$\sum_{v\in u} p(v,u,s) + \sum_{v\in u^c} p(v,u^c,s) \leq 1,$$

which means that either $u = s$, or $u^c = s$, or $s = u \circ_r \lambda(1)$, or $s = u^c \circ_r \lambda(1)$. We only need to consider terms in $\Delta_{q_1'q_2'}(\psi(x))$ corresponding to $u \subset s$, such that

$$\sum_{v\in u} p(v,u,s) + \sum_{v\in u^c} p(v,u^c,s) = 0,$$

which means that either $u = s$, or $u^c = s$. From this it follows that Ψ defines an equivalence mod $\hbar^2$ of the two deformations iff the degree 1 terms in t match, which is to say

$$\sum_{\{u\subset s | s=\lambda(1)\circ_r u\}} d_1 u\otimes\lambda(1) + d_2\lambda(1)\otimes u + \sum_w c_{s,w}\Delta(w) =$$
$$\sum_{\{u\subset s | s=\lambda(1)\circ_r u\}} c_1 u\otimes\lambda(1) + c_2\lambda(1)\otimes u + c_{s,w} w\otimes 1 + c_{s,w} 1\otimes w.$$

Of course $c_{s,w} w\otimes 1 + c_{s,w} 1\otimes w$ is the primitive part of $\sum_w c_{s,w}\Delta(w)$. For the equality we thus need $c_{s,w} = (c_1 - d_1)m = (c_2 - d_2)m_{su}$ if $w = \lambda(1)u$ for an u such that $s = \lambda(1)\circ_r u$ and m_{su} is cardinality of the orbit of the vertex $s - u$ under the automorphism group of s. Choose the other $c_{s,w}$ equal to 0. QED

7 The general approach in the associative case

The construction of bialgebras can be performed in much greater generality. Starting from an arbitrary Hopf operad P one can construct the operad $P[\lambda_n]$ which has as algebras P-algebras together with an n-ary operation. Under conditions similar to those in Theorem 4.2 one can find a Hopf-P algebra structure on the initial $P[\lambda_n]$-algebra (see [10], [11]). The explicit calculations for $P = \mathrm{Com}_*$, the operad of unital commutative algebras is presented in the previous sections. These can also be done for the operad Ass_* of unital associative algebras. In this section we briefly state some of the results.

7.1 Definition Let the category $\mathcal{A}_n$ of unital associative algebras with an n-ary map be the category which has as as objects pairs (A,α) of an associative unital algebra A and a linear map $\alpha : A^{\otimes n} \longrightarrow A$, and which has as maps $\mathcal{A}_n((A,\alpha),(B,\beta))$ the algebra homomorphisms $f : A \longrightarrow B$ such that $\beta\circ f^{\otimes n} = f\circ\alpha$. Let (A_n,λ_n) be the initial object in the category $\mathcal{A}_n$. Then A_n can be described as the free associative algebra on trees with edges coloured by $\{1,\ldots,n\}$ and at each vertex the incoming edges

of each colour endowed with a separate linear ordering (*planar n-trees*, for short).

7.2 THEOREM *The tensor algebra A_n on planar rooted trees with n-coloured edges has a natural family of graded connected Hopf algebra structures, indexed by sequences $(q_{11}, \ldots, q_{1n}, q_{21}, \ldots, q_{2n}) \in k^{2n}$. The inductive description of the coproduct is given by the formula of Theorem 4.2.*

The A_n together again form a simplicial algebra (cf. Remark 2.4).

7.3 COROLLARY *Let A_n be the free associative algebra on the planar rooted trees.*

(i). The coproducts on A_n of Corollary 4.2 are given by the closed formula

$$\Delta(t) = \sum_{s \subset t} \prod_j \left(\prod_{v \in s} q_{1\,j}^{p_j(v,s,t)} \cdot \prod_{v \in s^c} q_{2\,j}^{p_j(v,s^c,t)} s \otimes s^c \right),$$

where the product of trees is associative, non-commutative. The order of multiplication is given by the linear order on the roots of the trees defined by the linear on the incoming edges at each vertex and the partial order on vertices.

(ii). The vector space of primitive elements of the dual is spanned by elements dual to planar n-trees. The Lie bracket is the commutator of the (non-associative) product $\bullet$ given by

$$D_s \bullet D_t = \sum_w \sum_{s \subset w,\ s^c = t} \prod_j \left(\prod_{v \in s} q_{1\,j}^{p_j(v,s,w)} \cdot \prod_{v \in t} q_{2\,j}^{p_j(v,t,w)} D_w \right),$$

where w, s and t are trees with a linear ordering on the incoming edges of the same colour at each vertex and the inclusions of s and t in w have to respect these orderings.

PROOF The only change is that we have to remember the ordering of up going edges at each vertex. Than one can copy the proof of the commutative case verbatim. QED

7.4 REMARK Independently, Foissy [4] has found the formula for the Lie bracket in the case where $n = 1$, $q_{11} = 1$ and $q_{12} = 0$ (and Ass_* is the underlying operad). He uses this formula to give an explicit isomorphism between the Hopf algebras A_1 and A_1^* with this coproduct.

References

[1] C. Brouder and A. Frabetti. Renormalization of QED with Planar Binary Trees. *Euro. Phys. J. C*, 19(4):715–741, 2001.

[2] F. Chapoton and M. Livernet. Pre-Lie algebras and the rooted trees operad. *Internat. Math. Res. Notices*, 8:395–408, 2001. Also at `arXiv:math.QA/0002069`.

[3] A. Connes and D. Kreimer. Hopf Algebras, Renormalisation and Noncommutative Geometry. *Comm. Math. Phys.*, 199, 1998.

[4] L. Foissy. Les algèbres de Hopf des arbres enracinés décorés. II. *Bull. Sci. Math.*, 126(4):249–288, 2002. Also at `arXiv:math.QA/0105212`.

[5] M. Gerstenhaber and S.D. Schack. Bialgebra cohomology, deformations, and quantum groups. *Proc. Nat. Acad. Sci. U.S.A.*, 87(1):478–481, 1990.

[6] M.E. Hoffman. Combinatorics of rooted trees and Hopf algebras. *Trans. Amer. Math. Soc.*, 355(9):3795–3811 (electronic), 2003.

[7] D. Kreimer. On the Hopf algebra structure of perturbative quantum field theories. *Adv. Theor. Math. Phys.*, 2(2):303–334, 1998.

[8] J.-L. Loday and Maria O. Ronco. Hopf Algebra of the Planar Binary Trees. *Adv. Math.*, 139(2):293–309, 1998.

[9] J. Milnor and J. Moore. On the Structure of Hopf Algebras. *Ann. Math*, 81(2):211–264, 1965.

[10] I. Moerdijk. On the Connes-Kreimer Construction of Hopf Algebras. *Cont. Math.*, 271:311–321, 2001. Also at `arXiv:math-ph/9907010`.

[11] P.P.I. van der Laan. Operads - Hopf algebras and coloured Koszul duality. Ph.D. thesis Utrecht University, 2004.

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